Compensation of instabilities in magnetic Taylor-Couette flow

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The axisymmetric linear stability of the Taylor-Couette flow with an azimuthal magnetic field is considered. It is shown that a flow with the combination of a linearly unstable rotation and a linearly unstable azimuthal magnetic field can be linearly stable. The flow stabilization takes place for both ideal and dissipative flows. For dissipative flow the stabilization exists only for a combination of counter-rotating cylinders and a counterdirecting azimuthal magnetic field on cylinders. The effect can be important for the problem of a plasma confinement by the magnetic field.

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I. INTRODUCTION

The Taylor-Couette flow between concentric rotating cylinders is a classical problem of hydrodynamic and hydromagnetic stability [1,2]. According to the Rayleigh criterion, the ideal flow is stable to axisymmetric perturbations whenever the specific angular momentum increases outwards at every point of the flow,

$$\frac{d}{dR}(R^2\Omega)^2 > 0, \tag{1}$$

where the cylindrical system of coordinates (R,ϕ,z) is used, and Ω is the angular velocity.

In the presence of an azimuthal magnetic field, the necessary and sufficient condition for the axisymmetric stability of ideal Taylor-Couette flow is [3] (see also [4])

$$\frac{1}{R^3}\frac{d}{dR}(R^2\Omega)^2 - \frac{R}{\mu_0\rho}\frac{d}{dR}\left(\frac{B_\phi}{R}\right)^2 > 0, \tag{2}$$

where B_ϕ is the azimuthal magnetic field, ρ is the density, and μ_0 is the magnetic constant.

The rotation is called stable (unstable) if it fulfills (violates) the condition (1) [or equivalently if the first term of condition (2) is positive (negative)]. The azimuthal magnetic field is called stable (unstable) if the second term of condition (2) is positive (negative). Dissipative effects stabilize the flow. Ideally unstable rotation becomes really unstable only if an angular velocity exceeds some critical value. The same is true for an ideally unstable magnetic field which becomes really unstable also only if a magnetic field exceeds some critical value [5]. It has been shown [6] that the axisymmetric stability properties of the dissipative Taylor-Couette flow with an imposed azimuthal magnetic field can be classified just in accordance with the ideal condition (2). A combination of stable rotation and stable magnetic field is stable. The stable magnetic field stabilizes the unstable rotation (i.e., a critical angular velocity, at which flow becomes really unstable, increases with increasing magnetic field) and the unstable magnetic field destabilizes the stable rotation (and vice versa). The flow which has a combination of unstable rotation and unstable magnetic field is the more unstable (have smaller critical values) than both the unmagnetized flow with unstable rotation or a static unstable magnetic field [6].

Nevertheless, we will demonstrate below that the condition (2) guarantees the ideal flow stability for some combinations of unstable rotation and unstable magnetic field. For dissipative flow, our numerical results demonstrate the flow stability in some vicinity of ideal stability line.

II. BASIC EQUATIONS

Consider a viscous electrically conducting incompressible fluid between two rotating infinite cylinders in the presence of an azimuthal magnetic field. The equations governing the problem are

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P + \nu\Delta\mathbf{U} + \frac{1}{\mu_0}\mathrm{curl}\,\mathbf{B}\times\mathbf{B},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{U}\times\mathbf{B}) + \eta\Delta\mathbf{B},$$

$$\mathrm{div}\,\mathbf{U} = \mathrm{div}\,\mathbf{B} = 0, \tag{3}$$

where $\mathbf{U}$ is the velocity, $\mathbf{B}$ is the magnetic field, P is the pressure, ν is the kinematic viscosity, and η is the magnetic diffusivity.

For ideal flow ($\nu=\eta=0$), Eqs. (3) admit the solution in the cylindrical system of coordinates (R,ϕ,z):

$$U_R = U_z = B_R = B_z = 0,$$

$$B_\phi = B_\phi(R), \quad U_\phi = R\Omega(R), \tag{4}$$

where $\Omega(R)$ and $B_\phi(R)$ are arbitrary functions of radius fulfilling boundary conditions. For dissipative flow, the angular velocity and azimuthal magnetic field have fixed profiles

$$B_\phi = a_B R + \frac{b_B}{R}, \quad U_\phi = R\Omega = a_\Omega R + \frac{b_\Omega}{R}, \tag{5}$$

where a_Ω, b_Ω, a_B, and b_B are constants defined by the boundary conditions:

$$a_\Omega = \Omega_{\mathrm{in}}\frac{\hat{\mu}_\Omega - \hat{\eta}^2}{1-\hat{\eta}^2}, \quad b_\Omega = \Omega_{\mathrm{in}}R_{\mathrm{in}}^2\frac{1-\hat{\mu}_\Omega}{1-\hat{\eta}^2},$$

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$$a_B = \frac{B_{in}}{R_{in}} \frac{\hat{\eta}(\hat{\mu}_B - \hat{\eta})}{1 - \hat{\eta}^2}, \quad b_B = B_{in} R_{in} \frac{1 - \hat{\mu}_B \hat{\eta}}{1 - \hat{\eta}^2}, \tag{6}$$

where

$$\hat{\eta} = \frac{R_{in}}{R_{out}}, \quad \hat{\mu}_\Omega = \frac{\Omega_{out}}{\Omega_{in}}, \quad \hat{\mu}_B = \frac{B_{out}}{B_{in}}, \tag{7}$$

R_{in} and R_{out} are the radii, Ω_{in} and Ω_{out} are the angular velocities, and B_{in} and B_{out} are the azimuthal magnetic fields of the inner and outer cylinders, respectively.

Note that for viscous flow the magnetic field and angular velocity profiles are completely defined by the three parameters: $\hat{\eta}$, $\hat{\mu}_\Omega$, and $\hat{\mu}_B$. The first magnetic field term in Eqs. (5) corresponds to a constant axial electric current density into the fluid. The second term is current free.

We are interested in the stability of the basic solution (5). The axisymmetric linear stability problem is considered. By developing the disturbances into normal modes, solutions of the linearized magnetohydrodynamics equations are considered in the form

$$F = F(R)\exp[i(kz + \omega t)], \tag{8}$$

where F is all of the disturbances.

The dimensionless numbers of the problem are the magnetic Prandtl number Pm, Hartmann number Ha, and Reynolds number Re,

$$\mathrm{Pm} = \frac{\nu}{\eta}, \quad \mathrm{Ha} = \frac{B_{in} R_0}{\sqrt{\mu_0 \rho \nu \eta}}, \quad \mathrm{Re} = \frac{\Omega_{in} R_0^2}{\nu}, \tag{9}$$

where $R_0 = [R_{in}(R_{out} - R_{in})]^{1/2}$ is the length unit.

The detailed description of the equations and the numerical method used has been given in our earlier paper [5] and will not be reproduced here. Always no-slip boundary conditions for the velocity on the walls are used. The tangential electrical currents and the radial component of the magnetic field vanish on the conducting walls. The magnetic field must match the external magnetic field for the insulating walls [5].

There are some indications that the instability originates as a monotonic instability for the problem in hand. To the author's knowledge, the above statement has not been formally proved. Nevertheless, for the sake of simplicity, we take that $\omega = 0$ for the marginal stability lines below.

III. RESULTS

Let us demonstrate that the condition (2) permits the stability for some combination of an unstable rotation and an unstable magnetic field. Rewriting Eq. (2) in the form

$$R\frac{d\Omega^2}{dR} + 4\Omega^2 - \frac{R}{\mu_0 \rho}\frac{d}{dR}\left(\frac{B_\phi}{R}\right)^2 > 0, \tag{10}$$

we immediately see that the ideal flow is in particular stable if

$$\Omega = \pm \frac{1}{\sqrt{\mu_0 \rho}} \frac{B_\phi}{R}. \tag{11}$$

Let us recall that the angular velocity is the every function of radius, which fulfills the boundary conditions, for the ideal flow. So we can always choose Ω in such a way that Ω^2 increases with radius in some points (and the flow is obviously stable in these points) and Ω^2 decreases with radius so strong in another points that the flow is unstable. The flow is stable only if it is stable at every point. So the flow, which is stable in some points and unstable in others, is unstable by the definition. Obviously, the magnetic field, which is defined by Eq. (11), is stable (unstable) for points where Ω^2 decreases (increases) with radius. Like to rotation, the magnetic field, which is stable in some points and unstable in others, is unstable. Thus we have constructed the stable flow consisting of the ideally unstable rotation and ideally unstable magnetic field. Note, that the main feature of such combined stability is the stability of one component in the points where another component is unstable.

For dissipative profiles (5), we can write a more general relation

$$\frac{1}{\sqrt{\mu_0 \rho}} \frac{B_\phi(R)}{R} = \pm \Omega + C, \tag{12}$$

where

$$C = \pm \Omega_{in} \frac{\hat{\eta}\hat{\mu}_B - \hat{\mu}_\Omega}{1 - \hat{\mu}_B \hat{\eta}}, \tag{13}$$

and B_{in} is connected with Ω_{in} by the equality

$$\frac{1}{\sqrt{\mu_0 \rho}} \frac{B_{in}}{R_{in}} = \pm \Omega_{in} \frac{1 - \hat{\mu}_\Omega}{1 - \hat{\mu}_B \hat{\eta}}. \tag{14}$$

Using Eq. (12), the condition (2) takes the form

$$4\Omega^2 - 2RC\frac{d\Omega}{dR} > 0. \tag{15}$$

According to the conditions (1) and (2), the rotation and the magnetic field, which are defined by Eq. (5), are unstable if [4–6]

$$\hat{\mu}_\Omega < \hat{\eta}^2, \quad \text{and} \quad \hat{\mu}_B < 0 \quad \text{or} \quad \hat{\mu}_B > \frac{1}{\hat{\eta}}. \tag{16}$$

For unstable rotation $d\Omega/dR < 0$ and the condition (15) takes the form

$$\frac{\hat{\eta}\hat{\mu}_B - \hat{\mu}_\Omega}{1 - \hat{\mu}_B \hat{\eta}} > 2\Omega^2 \left(\Omega_{in} R \frac{d\Omega}{dR}\right)^{-1}. \tag{17}$$

The right-hand part maximum equals $-\hat{\mu}_\Omega$ for $\hat{\mu}_\Omega > 0$ and equals 0 for $\hat{\mu}_\Omega < 0$. Using these maximum values, it is easy to show that the condition (17) is fulfilled for the unstable rotation and the unstable magnetic field [see Eq. (16)] if

$$\hat{\mu}_\Omega < 0, \quad \hat{\mu}_B < 0, \quad \text{and} \quad \hat{\eta}|\hat{\mu}_B| \le |\hat{\mu}_\Omega|. \tag{18}$$

The flow is neutrally stable for $\hat{\eta}|\hat{\mu}_B| = |\hat{\mu}_\Omega|$.

To check the presence of stabilization for dissipative flow, we have made the numerical calculations for $\hat{\mu}_\Omega = -0.5$ and $\hat{\eta} = 0.5$. The insulating boundary conditions were used. Note that the axisymmetric mode is the most unstable one for chosen $\hat{\mu}_\Omega$ and $\hat{\eta}$ without the magnetic field [7]. The marginal stability lines are presented in Fig. 1. These lines do not

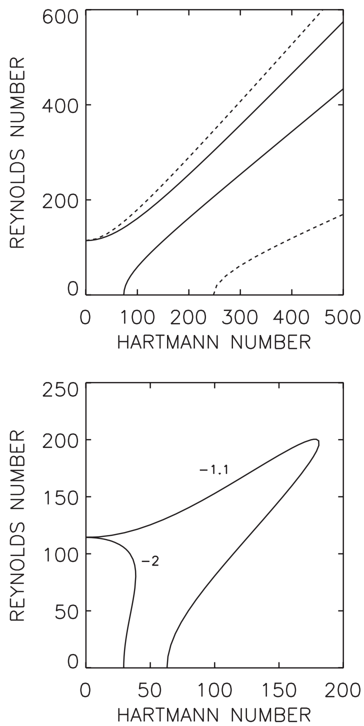

FIG. 1. The marginal stability lines for *insulating* cylinders with $\hat{\eta}$=0.5 and $\hat{\mu}_\Omega$=−0.5. On the left panel $\hat{\mu}_B$=−1 (solid lines) and −0.5 (dashed lines). The lines are labeled by the $\hat{\mu}_B$ values on the right panel. The flow is stable between the lines (left) and below the lines (right).

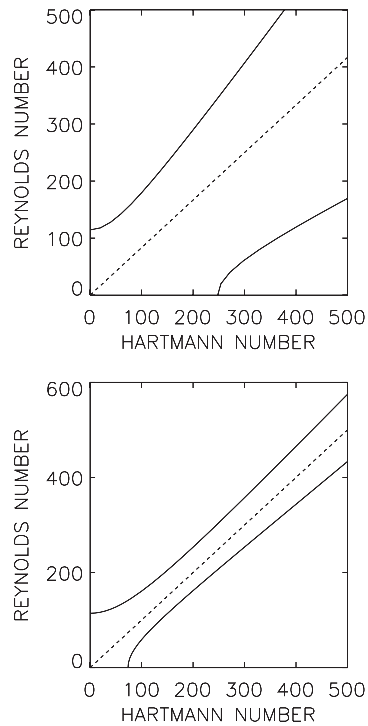

FIG. 2. The marginal stability lines for dissipative fluid (solid) and the ideal stability line according to Eq. (19) for *insulating* cylinders with $\hat{\eta}$=0.5, $\hat{\mu}_\Omega$=−0.5, and $\hat{\mu}_B$=−0.5 (left) and $\hat{\mu}_B$=−1 (right).

depend on Pm [5]. The stability region extends to infinitely large Reynolds and Hartmann numbers for combinations of unstable rotation and unstable magnetic field which fulfill the condition (18) [see left panel in Fig. 1] and restricts only finite Reynolds and Hartmann numbers for combinations of unstable rotation and unstable magnetic field which violates the last condition (18) (see right panel in Fig. 1 and [5,6]).

Figure 2 demonstrates that the stability region locates nearby to the line

$$\mathrm{Re} = \frac{(1-\hat{\eta}^2)^{1/2}}{\hat{\eta}^{1/2}} \frac{1-\hat{\mu}_B\hat{\eta}}{1-\hat{\mu}_\Omega}\mathrm{Ha}. \tag{19}$$

The relation (19) is exactly the relation (14) for ideal fluid which is expressed through dimensionless numbers (9) with Pm=1. The magnetic Prandtl number should be really taken as unity in a transition from dissipative to ideal flow when the viscosity and the magnetic diffusivity go to zero but their ratio is equal to unity. The dissipative effects lead to the flow stability not only exactly on the ideal stability line (19) but in some line's vicinity also.

IV. CONCLUSION

We have demonstrated that a combination of an unstable rotation and an unstable azimuthal magnetic field can generate a stable Taylor-Couette flow. The stabilization can take place only if both the rotation profile and the magnetic field profile are partly unstable. The profile is called partly unstable if the stability condition fulfills in some profile's points and violates in another profile's points. Then, the unstable parts of one component can be stabilized by the stable parts of another component and vice versa.

For ideal flow the combination of the unstable rotation and the unstable magnetic field can become stable only for some particular value of the magnetic field for a given angular velocity value. For dissipative flow the stability takes place in some vicinity of the ideal flow stability (see Fig. 2) due to the dissipative effects. Moreover, for dissipative flow the combination of an unstable rotation and an unstable azimuthal magnetic field can become stable only if both the angular velocity and the azimuthal magnetic field change the sign and $\hat{\mu}_B$ is smaller by absolute value than $\hat{\mu}_\Omega$ [see Eq. (18)].

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